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ON THE EXISTENCE OF SIMULTANEOUS EDGE DISJOINT REALIZATIONS OF DEGREE SEQUENCES WITH 'FEW' EDGES

D. J. Kleitman, et al

Cornell University

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August 1975

ON THE EXISTENCE OF SIMULTANEOUS EDGE DISJOINT REALIZATIONS OF DEGREE SEQUENCES WITH "FEW" EDGES

by

D.J. Kleitman*
Massachusetts Institute of Technology

M. Koren**
University of Pennsylvania

S.-Y. Li
Massachusetts Institute of Technology

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ABSTRACT

This paper contains the following results. For a sequence A, let M_A be the number of non-zero entries in it; suppose A, B and $C = (A+B)$ are sequences that are the degrees of simple graphs; then if $\sum c_i \leq 2(M_A + M_B - 2)$, there exists a realization of C having disjoint factors with degree sequences A and B. At least one of these can be a forest. If A and B are forest realizable, the conditions under which A, B and C can be simultaneously realized with A- and B-factors that are forests are characterized.

1. INTRODUCTION

Let $A = \{a_i\}$ and $B = \{b_i\}$ represent sequences of length n of integers, and suppose that there exist (simple) graphs G_1, G_2, G_3 on n vertices such that the degree of the jth vertex in G_1 is a_j, in G_2 is b_j, and in G_3 is $a_j + b_j$. The sequences A, B and $C = \{a_i + b_i\}$ are then said to be "realizable sequences".

A number of authors have considered the question of the simultaneous realization of such sequences; that is, of the existence of a graph G_3 realizing C having an "A-factor" (or subgraph having degree sequence A), (whose complement in C is then a B-factor.) S. Kundu showed that if all the entries of B are k or $k + 1$ then such a realization exists [5]; he also showed [6] that if A and B are "tree realizable" (b_j, $a_j \geq 1$, $\Sigma a_j = \Sigma b_j = 2n - 2$) then the same result holds, with the realizations of A and B both trees. One of the present authors (M.K.) showed that the result holds if A and B are "forest realizable" ($\Sigma a_j \leq 2(M_A-1)$, $\Sigma b_j \leq 2(M_B-1)$ where M_X is the number of nonzero index entries in X), though it may not be possible for the realizations of A and B both to be forests [4]. On the other hand he showed by an example that the condition $\Sigma(a_j + b_j) = 4(n - 1)$ is not sufficient to guarantee that simultaneous realizations exist [4].

It is the purpose of this paper to show that the condition

$$\sum_{j=1}^{n} (a_j + b_j) \leq 2(M_A + M_B - 2) \tag{1}$$

does guarantee the existence of simultaneous realizations, and to provide a precise characterization of the sequences for which A and B are forest realizable yet there is no simultaneous realization of A, B, and C with A and B forests.

Very little is known about realizable sequences A, B and C that are not simultaneously realizable. Koren has characterized those for which A is a "claw" [3]; it may well be that while such sequences are possible for $\Sigma(a_j + b_j) > 2(M_A + M_B - 2)$ they may occur for only a very restricted class of sequences, and characterization of that class may well be possible.

2. THEOREMS AND DISCUSSION OF PROOFS

Our main result is:

Theorem 1: Let $A = \{a_i\}$, $B = \{b_i\}$, $C = \{a_i + b_i\}$ be realizable sequences of length n, for which (1) holds. Then A, B, and C are simultaneously realizable. At least one of A or B can be a forest in a simultaneous realization.

We prove this by first noting that (if A, B and C are nontrivial) there must be an index for which a and b are respectively (1,1),(1,2),(2,1),(1,0) or (0,1).

We then, when there are 6 or more vertices, give a prescription for taking such sequences and reducing them to shorter sequences having the same properties, or to sequences for which Kundu's factor theorem may be applied. When n is five or less the graphs may be explicitly determined.

The prescription is basically this. A vertex v_j that is to have degrees a_j and b_j of (1,0) is to be connected in A (and C) to a vertex of largest degree in A; similarly with A replaced by B for degrees (0,1). This connection is not always possible, since it may lead to a sequence C' which is not realizable.

We shall, in section 3, indicate the exact circumstances in which this failure can happen and, in section 4, exhibit a resolution. If there exists (1,1) vertex but no (1,0) vertex, the (1,1) vertex will be split into two and hence the sequences are reduced to the (1,0) case. This is done in section 5. Finally when no (1,0),(0,1) or (1,1) vertices exist, we modify the sequences by replacing a (1,2) vertex by (1,0). Then we show how to construct realizations of the original sequences from realizations of the modified sequences.

The same procedure yields as a second conclusion that if A and B are forest-realizable then A, B and C can be simultaneously realizable with A and B forests if and only if certain conditions hold:

Theorem 2: Let A, B, C be realizable sequences as above with A and B forest-realizable. Then C is realizable with two disjoint forest factors having degrees A and B if and only if at least one of the following conditions holds.

a) Neither A nor B is a "claw" with at least one isolated vertex. (By a claw we mean that one vertex meets every arc.)

b) $a_i \quad \Sigma 2(M_A - 1)$ or $\Sigma b_i \quad 2(M_B - 1)$.

c) A is a claw and

c_1) There is an index j with $a_j = 0$ and $b_j > 1$,

or c_2) B is a claw with the same 'center" as A.

or c_3) If $a_k > 1$ then $b_k = 0$.

d) The same as condition c, with interchanged roles for A and B.

If none of these conditions hold, it is easily seen that the non-claw sequence (A, say) can only be realized with a cycle, since $\Sigma a_i = 2(M_A - 1)$, and any A-factor will have at least two non-trivial components. In fact, A then can be realized with exactly one cycle. The converse will be proven through our procedure. The same arguments will also show the following:

Corollary: If A is not forest-realizable, then C has a realization in which the B-factor is a forest, and the A-factor is connected except possibly for isolated vertices.

3. CRITICAL SEQUENCES

Erdős and Gallai showed [1] that a sequence C: $c_1 \geq c_2 \geq \dots \geq c_n$ of nonnegative integers, with even sum, is realizable iff, for $k = 1,\dots,n$.

$$\sum_{i=1}^{k} c_i - k(k-1) \leq \sum_{i=k+1}^{n} \min(c_i, k). \qquad (2)$$

From now on, let us suppose that $C = A + B$ is nonincreasing. It is easy to see, that if $c_n = 1$ ($a_n = 1$, $b_n = 0$, say) and if for each k, the right-hand side of (2) exceeds the left-hand side at least by two, then we may connect v_n in A and in C to a vertex of largest degree in A.

We call a sequence <u>k-critical</u> if the right-hand side exceeds the left by at most 1. We call the sequence <u>completely k-critical</u> or <u>partially k-critical</u> depending on whether the difference is 0 or 1.

If a sequence is partially k-critical, no vertex of degree k or less may be connected to another such vertex by an arc in any realization, but if there are no other more restrictive criticalities, any other vertex may be linked by an arc to any vertex in some realization of C.

If a degree sequence is completely k-critical, then no vertex of degree k or less can be linked by an arc to any vertex other than the first k.

In the next lemma, we summarize those properties of critical sequences which also fulfill condition (1), which we will need for our proof.

<u>Lemma 3</u>: If $C = c_1 \geq \cdots \geq c_n$ fulfills condition (1) and is k-critical then:

a) $k \leq 4$.

b) If $k = 2$, then at most three of the a_i, b_i, $i \geq 3$ may be larger than 1.

c) If $k = 3, 4$ then $a_i, b_i \leq 1$ for $i > 4$.

<u>Proof</u>: Let Z, Y, X and W represent the number of vertices v_i, $i > k$ of the form $(1,0),(1,1),(\ell,0),(\ell,\ell'')$, respectively, with $\ell \geq 2$, $\ell'' \geq 1$. (in this lemma, the pairs are unordered). Let G realize C.

Since C is critical, there are at least $\min(c_i, k)$ edges in G between the vertex v_i $(i > k)$ and the set $\{v_1, \ldots v_k\}$ with at most one exemption, if the criticality is partial. Hence:

$$\frac{1}{2} \sum c_i \geq \frac{1}{2}k\,(k-1) + Z + 2Y + 2X + W \cdot \min(3,k) - 1.$$

Note that the -1 appears only if C is partially k-critical, and then there is at least one index $i > k$ such that $c_i > k$. On the other hand,

$$M_A + M_B - 2 \leq 2k + Z + 2Y + X + 2W - 2$$

where the $2k$ accounts for the possibility that $a_i, b_i \geq 1$ for $i = 1,\ldots,k$. Thus, by condition (1),

$$\frac{1}{2}k\,(k-1) + X + W - 1 \leq 2k. \qquad (3)$$

The proof follows by substituting, in (3) appropriate values for k.

4. THE (0,1) AND (1,0) CASES

The proof of Theorem 1 is by induction.

We can verify our induction hypothesis by explicit construction in all cases for five or fewer vertices. We therefore assume that we have six or more vertices. For simplicity we assume that the degree sequence B is forest-realizable (while A may or may not be).

It is a result of Kundu that if all degrees of B (or A) are zeroes or ones then there exists a realization of C with A and B factors. We may therefore restrict our intention to sequences for which at least one degree of A and one of B are two or more.

In this section we assume $c_n = 1$. We will examine five cases.

4.1. $a_n = 0$ AND C 1-CRITICAL

We first assume that C is completely one-critical. Then $a_1 + b_1 = n - 1$. Call the vertex of highest B degree v_B. Then if $b_B < M_B - 1$ and $b_1 > 1$, we may connect v_n to v_1 rather than to v_B, and all conditions of our theorem will hold on the remaining vertices. Recall that sequence B is assumed to be forest-realizable.

Suppose $b_B = M_B - 1$; since $a_n = 0$, we have $M_A \leq n - 1$. Thus the $(n-1)$ edges touching v_1 and the $(M_B - 2)$ other edges touching v_B will account for all $M_A + M_B - 2$ edges, and we must have $M_A = n - 1$, and hence $a_B = 1$. But if $a_B = 1$, we have $c_1 = n - 1 \geq M_A$ and $c_2 = M_B$, which imply the existence of at least $M_A + M_B - 1$ edges, a contradiction.

Suppose instead that $b_B < M_B - 1$, but $b_1 = 1$; then the degrees of vertex v_1 are $(n-2, 1)$. We connect v_1 to every vertex in A except v_n which we connect it to in B. The remaining degree sequences A, B and C obviously will all be realizable. We need only show that condition (1) still holds after these connections are made. The left-hand side, after these connections diminishes by $2(n-1)$ or at least $2M_A$. If any part of A remains to be constructed, the new $(M_A - 1)$ is at least one, while M_B has diminished by 2; the right-hand side has diminished by at most $2((M_A - 2) + 2)$ so that the inequality still holds. If A is entirely constructed, one can simply realize B and the inequality is irrelevant.

The case that C is partially one critical is similar.

Let us say that a sequence is <u>super-k-critical</u> if it violates condition (2), for k. We will treat now the "symmetric" case:

4.2. $b_n = 0$ AND C 1-CRITICAL

The same arguments hold except that A may not be forest-realizable. Call the vertex of highest degree v_A. It is necessary to verify that if

$a_1 + b_1 = n - 1$, $a_1 > 1$, and $a_A < M_A - 1$, then connecting v_n to v_1 does not make A super-k-critical for $k \geq 2$. We may assume that A is originally k-critical for some $k \geq 2$. The number of arcs of A other than those connecting to v_1 must be at least $M_A - 2$ and equal to it only if all A degrees except three are ones. The total number of arcs in C would then necessarily be at least $M_A - 2 + a_1 + b_1 = n + M_A - 3$. Since $M_B \leq n - 1$, this bound is $M_A + M_B - 2$ which must therefore be an equality. This implies $M_B = n - 1$, $b_1 = n - 2$, $a_1 = 1$, in this case both A and B are tree realizable sequences and B is a claw so that its realization is trivial and A may be realized from the remainder as in the previous case.

The cases with higher criticalities are simpler:

4.3. C 4-CRITICAL

There is no problem here since all connections will in our procedure automatically be made toward the four largest degree vertices, avoiding 4-super-criticality.

4.4. C 3-CRITICAL

This case presents a problem of supercriticality only if the suggested connections of v_n involve a vertex other than v_1, v_2 or v_3. Since the connections indicated go to vertices with a_i or $b_i \geq 2$, the only way this could happen would be if a_4 (say) $= 2$; again this would be a problem only if $a_1 = a_2 = a_3 = 1$ and $b_n = 0$. In either case, if the sequence B is to avoid super-criticality, we must have $a_i = 0$ for $n = i > 4$, and as already noted $b_i \leq 1$ for $n \neq i > 4$.

We may therefore consider a vertex v_i, and treat it, rather than v_n and avoid this problem, unless there are no such vertices, so that $n = 5$.

4.5. C 2-CRITICAL

As in the previous case, trouble can arise only if (say) $a_1 = a_2 = 1$ while (say) $a_3 = 2$. If this should occur, then B non-super 2-criticality implies that $(a_3, b_3) = (2,1)$, $(a_4, b_4) = (1,2)$ and $a_i = 0$ for $n \neq i > 4$. Again, this case may be avoided by connecting v_i for $n \neq i > 4$ before v_n unless $n = 5$.

We have shown how to proceed with the induction hypothesis if $c_n = 1$. In order to finish the proof we show in the next two sections how to reduce the other cases into this one.

5. THE (1,1) CASE

If $(a_n, b_n) = (1,1)$, we replace v_n by two vertices $v_n' \sim (1,0)$ and $v_n'' \sim (0,1)$; that is, we change A into $A' = (a_1,\ldots,a_n,0)$ and B into $B' = (b_1,\ldots,b_{n-1},0,1)$. These changes do not destroy the realizability of A', B' or C' $(= A' + B')$, and, of course, condition (1) holds for A', B' and C'.

Let G' be a realization of C', having an A'-factor G_A' and a B'-factor $G' - G_A' = G_B'$. If v_n' and v_n'' have no common neighbor in G', we obtain a desired realization G of C simply by contracting them into a vertex v_n.

Suppose that $(v_n', v_r) \in G_A'$ and $(v_n'', v_r) \in G_B'$. In this case, since $c_r < (n+1) - 1$, we may choose a vertex v_s, such that $(v_s, v_r) \notin G'$. Let $(v_s, v_t) \in G'$, say, $(v_s, v_t) \in G_A'$. In G_A' we may replace the edges (v_n', v_r), (v_s, v_t) by (v_n', v_t), (v_r, v_s). After this "interchange", we can contract v_n' and v_n'', as before.

For the proof of Theorem 2, we have now to show how to modify the contraction, when the interchange introduces new cycles into G_A. This may happen only when there is a path in G_A' between v_s and v_r, and v_t is not connected by a path in G_A' to any vertex of the path except through v_s. In this case, we have a cycle in G_A (after the contraction) which passes through (v_r, v_s). If

$(v_r, v_t) \notin G$, then, since $(v_n, v_s) \notin G$, we may apply a $(v_r, v_s), (v_n, v_t)$ interchange, and "destroy" the cycle. Suppose, therefore, that $(v_t, v_r) \in G_B$. Let v_j be the neighbor of v_s in the cycle, other than v_r. Since the interchange and contraction exclude (v_t, v_s) from G, and since $(v_n, v_j) \notin G$, the interchange from $(v_n, v_t), (v_s, v_j)$ to $(v_n, v_j), (v_s, v_t)$ in A removes the cycle.

In the next section we assume that there are no (1,0), (0,1) or (1,1) vertices, and hence there are (1,2) or (2,1) vertices.

6. THE (2,1) OR (1,2) CASE

The connection procedure involved here is to reduce the degrees of a vertex v_j for example from (2,1) to (0,1). The idea here is that if the remaining sequences are all realizable then we can realize them and then insert the vertex v_j on any arc in A in the realization connecting two vertices other than the one to which v_j gets connected to in B.

It is clear that this degree change preserves condition (1), and doesn't effect B or its realizability at all. If follows from Lemma 2, from the assumption $n > 6$ and from the assumption that there are no (1,0), (0,1) or (1,1) vertices, that the realizability of C is not affected. Thus we can employ this procedure so long as

1. It does not destroy A realizability, and
2. We are assured that after the realization, the vertex v_j can be reconnected, which will follow if no vertex of A touches every one of A's edges.

We will show that changing $a_j = 2$ to $a_j = 0$ cannot effect the realizability of A here; and that reconnecting v_j can produce no difficulty. Note that if there are no (1,0), (0,1) or (1,1) vertices, then by condition (1), there are at least four vertices from the set {(2,1), (1,2)}. Therefore, we may assume that at least two vertices have degrees (2,1).

The first result is the following lemma

Lemma 4: Let $C = A + B$ be a degree sequence having no degrees (0,1), (1,0) or (1,1) and satisfying condition (1) with at least two vertices having degrees (2,1) with A, B, and C realizable. Then if the degrees of v_j are altered from (?,1) to (0,1) the new degree sequence A is still realizable.

Proof: Otherwise, the new A would violate some Erdös-Gallai condition; now the number of arcs in B is at least $M_B - s/2$ where s is the number of vertices having degree one in B. Each vertex counted by s must have degree 2 or more in A.

Suppose the $k = 1$ Erdös-Gallai condition were violated; then $a_1 = M_A-1$, and there must have been at least $M_A - 1 + s/2$ arcs in A, M_A-1 connecting to v_1 and at least $s/2$ others. This violates condition (1). If the $k = 2$ or higher Erdös-Gallai conditions were violated, the number of arcs in A would be at least $M_A + s - 2$ and s is at least 2, which still violates (1) (for example for $k = 2$, after the change counting one arc for the second vertex one arc for every vertex after the second and a second arc for every vertex after the second having degree 2 or more (at least $\min\{s - 2,0\}$ of these), the sequence would have to be 2-supercritical; the total number of arcs in A is at least greater than $M_A + s - 3$ or if $s \leq 2$ than M_A.

We now consider the final condition to be proven.

Lemma 5: With the hypothesis of Lemma 4 either no vertex in the "new" A meets all A's edges, or the same is true in B, for $n > 4$.

Proof: If a vertex in A was to meet every arc of A, then $a_1 = M_A-1$. If $M_A - 1 \geq 3$, the second vertex having degrees (2,1) will have an edge passing through it in A that fails to meet v_1, as desired. If $M_A-1 = 2$, then that vertex could be v_1, which would mean that the original A degree sequence was $\{2,2,1,1,0,0,\ldots,0\}$. The original B degree sequence would have to be at least $\{1,1,2,2,2,\ldots\}$ since $n > 4$, and by interchanging A and B here we can obtain the desired result.

We have thus indicated how our procedures apply in all cases, and the proof is complete.

We notice, that every step of the construction except one always preserves the tree-like or forest-like nature of what is constructed. Connecting vertices of degree 1 in A or B of course maintains that structure; but in addition, the procedures above assure that the remaining degree sequences are tree-like since every edge is connected to vertex of at least 2 except in case 4.1 when $a_1 = 1$.

For a degree sequence A to be forest-realizable, it must have a number of arcs no greater than M_A-1.

Therefore if A and/or B are forest-realizable, the construction procedure upon iteration will yield realizations that are forests, unless at some stage case 4.1 (or 4.2) occurs; at that stage C is critical, with $a_1 = 1$, and $C = (1,0)$. This means that $b_1 = n - 2$, and B is 1-critical also.

It is easy to see that this configuration can only arise as intermediate stage after only step 4.1 and step 4.2 edge assignments have been made. It will lead to a realization that is not a forest for A if and only if the number of arcs of A is M_A-1, since after the connection of v_1, M_A is then reduced by two but the number of A edges by only one.

For this to occur, the original A, B and C must have had a vertex v_1 critical in B; must have had no vertex having degree 2 or more in A and degree 0 in B; and must satisfy (1) as an equality. These are the conditions of Theorem 2.

Obviously the same remarks hold for A and B interchanged. If A is not forest-realizable, then this can never happen for B and the realization of B will always be forest-like. If this case occurs, it obviously occurs only for one of A and B, and the realization of the other will be a forest.

This completes the proof of Theorems 1 and 2. Similar arguments prove the corollary to Theorem 2.

7. CONCLUSIONS

The arguments above can fail in a number of different ways when condition (1) is not satisfied. Some of these lead to classes of degree sets A, B, C, that are not simultaneously realizable, though each of A, B, and C is. It is possible that the nature of such sets can be characterized through considerations of this kind.

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